Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (9 May, 2016)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the* 2 *PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks.*
- *The answer to each question is a number, function, set, inequality, random variable etc. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*

Q.1 Consider the vector space $\mathbb{R}^{2016}$ over the field $\mathbb{R}$ of real numbers. What is the smallest positive integer k for which the following statement is true: given any k vectors $v_1, \ldots, v_k \in \mathbb{R}^{2016}$, there exist real numbers $a_1, \ldots, a_k$, not all zero, such that $a_1v_1 + \cdots + a_kv_k = 0$ and $a_1 + \cdots + a_k = 0$.

Q.2 Find all complex triples (a, b, c) such that the following matrix is diagonalizable

$$\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Q.3 For what values of k does the linear system

$$x - 3z = -3$$

$$2x + ky - z = -2$$

$$x + 2y + kz = 1$$

in unknowns x, y, z have no solution?

Q.4 Let V and W be subspaces of the vector space $\mathbb{R}^9$ over the field $\mathbb{R}$ of real numbers with $\dim V = 5$ and $\dim W = 6$. Then what is the smallest possible dimension of $V \cap W$?

Q.5 Consider the inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = 2a_1b_1 - a_1b_2 - a_2b_1 + 5a_2b_2$$

on $\mathbb{R}^2$. Write down a vector which is orthogonal to $(1, 0)$ and has norm 1.

Q.6 Find all values $\alpha \in \mathbb{R}$ for which the matrix

$$\begin{bmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

is positive definite.

Q.7 Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ be a real matrix and c_1, c_2, c_3 be the scalars

$$c_1 = \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right), \quad c_2 = \det\left(\begin{bmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{bmatrix}\right), \quad c_3 = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right).$$

Find all triples (c_1, c_2, c_3) so that rank $A = 2$.

Q.8 Let A be the complex 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}.$$

Find all triples (a, b, c) for which the characteristic and minimal polynomials of A are different.

Q.9 Given $\epsilon > 0$, what is the largest δ which fits the definition of continuity of the function

$$f(x) = \begin{cases} \frac{x+3}{2} & \text{if } x \leq 1 \\ \frac{7-x}{3} & \text{if } 1 \leq x \end{cases}$$

at $x = 1$, that is, the largest $\delta > 0$ for which the implication $|x - 1| < \delta \implies |f(x) - f(1)| < \epsilon$ holds?

Q.10 A point ω is said to be a *fixed point* of a function f if $f(\omega) = \omega$. Given that the function

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points α, β, γ in $(-2, -1)$, $(0, 1)$ and $(1, 2)$ respectively, let us define a sequence of real numbers $\{x_n\}$ as

$$x_1 = \gamma - 0.01\ , \ x_{n+1} = f(x_n) \quad n = 1, 2, 3, \ldots.$$

Given that the sequence converges, find

$$\lim_{n\to\infty} x_n.$$

Q.11 Suppose f is a real valued continuously differentiable function on $[0,1]$ with $f(0) = f(1) = 0$ and

$$\int_0^1 f^2(x)\,dx = 1.$$

Find the value of $\int_0^1 xf(x)f'(x)\,dx$.

Q.12 Let

$$y_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}.$$

Find $\lim_{n\to\infty} y_n =?$

Q.13 Find all values of $x > 0$ for which the series

$$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots + \frac{n^nx^n}{n!} + \cdots$$

converges.

Q.14 Find all values of (p,q) for which the integral

$$\int_0^1 x^p\ \log^q(1/x)\,dx$$

converges.

Q.15 Suppose X, Y, Z are i.i.d. Uniform$[0,1]$ random variables. What is the probability $\mathbb{P}(XY < Z^2)$? Write down your answer as a fraction.

Q.16 Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables for which $\mathbb{E}(X_1^{-1}) < \infty$, where $\mathbb{E}$ denotes expectation. Let $S_i := X_1 + X_2 + \cdots + X_i$. For $m < n$ calculate $\mathbb{E}(S_m/S_n)$.

Q.17 Suppose the distribution of Y, conditional on $X = x$, is Poisson(x) and random variable X is exponentially distributed with rate parameter 1. Find the correlation between X and Y.

Q.18 Let $U_1, U_2, \ldots, U_n$ be i.i.d. Uniform$(0, \theta)$, $\theta > 0$ random variables. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ^2.

Q.19 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with one of two possible probability density functions $f(x|\theta)$. If $\theta = 0$, then $f(x|\theta) = I_{(0,1)}(x)$ while if $\theta = 1$, then $f(x|\theta) = \frac{1}{2\sqrt{x}} I_{(0,1)}(x)$. Find the maximum likelihood estimator $\hat{\theta}$ of θ.

Q.20 Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed known constants, and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma^2)$, σ^2 known. What is the distribution of MLE of β?