Department of Mathematics, IIT Bombay
Screening Test for PhD Admissions (May 10, 2018)
Time allowed: 2 hours and 30 minutes

Departmental Reg No. : **Maximum Marks: 40**

Name : **Choice:** | Math | Stat |

Instructions :

- Write your name and registration number in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs.
- All questions carry 2 marks. If there are 2 parts to a question, then each part carries 1 mark.
- There will be no partial credit. Simplify all your answers. In particular, the answer should not be in the form of a sum or product.
- The answer to each question is a number, function or a set.
- Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions.
- You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.
- If your marks in the written test is ≥ 16, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.

Probability - 6 Questions

1. Suppose we have a random sample of size n from the probability density function $f(y) = 1/2$, $0 < y < 2$. Find the moment generating function of $\sum_{i=1}^{n} Y_i$ at $t = 1/2$.

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2. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with mean 2 and variance 1. Take $Y_n = (X_{2n-1} + X_{2n})^2$, $n = 1, 2, \ldots$ and let $Z_n = \sum_{i=1}^{n} Y_i/n$. The constant c to which Z_n converges with probability 1 as $n \to \infty$ is

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3. Let X_1, X_2 and X_3 be independent random variables with common mean μ and common variance σ^2. Define $U = X_1 + X_2 + X_3$ and $V = X_1 + X_2 - 2X_3$. The correlation coefficient between U and V is

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4. Suppose X_1 and X_2 are independent Poisson random variables with mean θ. Given $X_1 + X_2 = 2$, the conditional expectation of X_1 is

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5. A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, he needs to answer all three questions correctly. What is the probability that the student will pass the examination if he knows the correct answers to 90 questions on the list?

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6. The pdf of Y, where $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$ for $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$ is

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Linear Algebra - 8 Questions

7. Consider the vector space $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) equipped with the inner product

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = 2(a_1b_1 + a_2b_2 + a_3b_3) - (a_1b_2 + a_2b_1 + a_2b_3 + a_3b_2).$$

Find all vectors in $\mathbb{R}^3$ of norm 1 which are orthogonal to the plane $x_1 - 2x_2 + 2x_3 = 0$.

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8. Let $A = \begin{bmatrix} 0 & 4 & 1 & -2 \\ -1 & 4 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 0 \end{bmatrix}$.

The minimal polynomial of A is

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9. For each integer $j \geq 1$, let V_j denote the real vector space of all polynomials in two variables of degree strictly less than j.

   (a) The dimension of V_{100} is

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   (b) The dimension of the space of all linear maps from V_5 to V_{11} whose kernel contains V_3 and whose image is contained in V_7, is

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10. Let A be the complex square matrix of size 2018 whose diagonal entries are all -2018 and off-diagonal entries are all 1.

   (a) The eigenvalues of A are

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   (b) What are the geometric multiplicities of the eigenvalues of A?

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11. Consider the vector space $V = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_{11} x^{11} : a_i \in \mathbb{R} \right\}$.

   Define a linear operator A on V by $A(x^i) = x^{i+4}$ where $i + 4$ is taken modulo 12.

   Find (a) the minimal polynomial of A and (b) the characteristic polynomial of A.

   (a) ——————————— (b) ———————————

12. Let A be a diagonal matrix whose characteristic polynomial is

$$P(x) = (x-16)^8 (x-15)^8 (x-14)^7 (x-13)^7 (x-12)^6 (x-11)^6 \cdots (x-2)(x-1).$$

   Let V be the set of all 72×72 matrices commuting with A. Let W be the subset of V consisting of all diagonal matrices.

   (a) The dimension of W is ———————————

   (b) The dimension of V is ———————————

13. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 4 & 4 \\ 1 & 4 & 27 & 7 \\ 1 & 4 & 7 & 52 \end{pmatrix}$.

   It is given that $A = GG^T$, where G is a 4×4 lower triangular matrix with positive diagonal entries, and G^T denotes the transpose of G.

   Find (a) the smallest and (b) the largest eigenvalue of G.

   (a) ——————————— (b) ———————————

14. Consider the matrix $A_\alpha = \begin{pmatrix} 4 & 2 & 2 \\ 2 & \alpha^2 + 4\alpha + 5 & \alpha + 3 \\ 2 & \alpha + 3 & -4\alpha^2 + 4\alpha + 14 \end{pmatrix}$.

   Find the set of all real numbers α such that A_α is positive definite.

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Real Analysis - 6 Questions

15. (a) All real numbers p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges are

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(b) All real values of x for which the series $\sum_{n=2}^{\infty} \frac{x^n}{n(\log n)^2}$ converges absolutely are

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16. Let $f_1 : [-1, 1] \to \mathbb{R}, \quad f_1(0) = 0$ be a continuously differentiable function and $\lambda > 1$. Consider the sequence of functions defined inductively by

$$f_k(x) := \lambda f_{k-1}(x/\lambda), \qquad k \geq 2, \quad x \in [-1, 1].$$

The pointwise limit of the sequence of functions (f_n) is

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17. Find the exact number of real roots of the following polynomials:

(a) $x^3 + 50x + 35$. (b) $x^3 + 50x - 105$.

(a) _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ (b) _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

18. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{n + x^3 \cos x}{ne^x + x^5 \sin x}$, $n \geq 1$. Find $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$.

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19. Evaluate: $\displaystyle\int_0^1 x^5 \left\{\log\left(\frac{1}{x}\right)\right\}^3 dx$.

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20. You can use the Fourier sine series of 2π-periodic odd function $f(x) = \frac{1}{8}\pi x(\pi - x)$ for $x \in [0, \pi]$.

The sum of the series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ is

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