Department of Mathematics, IIT Bombay
Screening Test for PhD Admissions (December 4, 2018)
Time allowed: 2 hours and 30 minutes

Departmental Reg No. : **Maximum Marks : 40**

Name : **Choice :** | Math | Stat |

Instructions :

- Write your name and registration number in the blank space, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs.
- All questions carry 2 marks. If there are 2 parts to a question, then each part carries 1 mark.
- There will be no partial credit. Simplify all your answers and if your answer is a fraction, then do not convert it into a decimal number.
- The answer to each question is a number, function, matrix or a set.
- Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions.
- You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.
- If your marks in the written test is ≥ 16, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.
- For a function f, f' denotes its derivative and for a matrix A, A^T denotes its transpose.

Probability - 6 Questions

1. Suppose the distribution of a random variable Y, conditional on $X = x$ is Normal (x, x^2) and the marginal distribution of X is Uniform $(0, 1)$. Then $\text{Var}(Y)$ is

   Ans. --

2. Let X be a random variable with probability density function $f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$, and let $A = \{X \geq 1\}$. Then $E\big(X \big| A\big)$ is

   Ans. --

3. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables such that X_1 is a Poisson (1) random variable. Let $Y_n = X_1^2 + X_2^2 + \cdots + X_n^2$, $n \geq 1$. Then $\lim_{n\to\infty} P\{Y_n > 2n\}$ is

Ans. --

4. Let $X_1, X_2, \cdots$ be independent and identically distributed uniform (0, 1) random variables and N denote the smallest integer $n \geq 1$ such that $X_1 + X_2 + \cdots + X_n > \frac{1}{2}$. Then $P(N > 3)$ is

Ans. --

5. Consider two coins, an unbiased coin (i.e. probability of head $\frac{1}{2}$) and a biased coin with probability of head $\frac{1}{3}$. A coin is selected at random and the toss resulted in a head. The probability that the selected coin is unbiased is

Ans. --

6. Let $X_1, X_2 \cdots, X_{100}$ be positive identically distributed random variables and let $S_n = X_1 + X_2 + \cdots + X_n$ for $n = 1, 2, \cdots, 100$. Then $E\left[\dfrac{1}{S_{100}} \displaystyle\sum_{k=1}^{100} 2S_k\right]$ is

Ans. --

Linear Algebra - 8 Questions

7. (a) Let V be a real vector space and $v_1, \ldots, v_{16} \in V$. Assume that $\displaystyle\sum_{i=1}^{8} a_{2i-1}v_{2i-1} = 0$ has infinitely many solutions and $\displaystyle\sum_{i=1}^{8} a_{2i}v_{2i} = 0$ has a unique solution. Find the maximum possible dimension of $W := \text{Span}\{v_1, \ldots, v_{16}\}$.

   (b) Let S be the real vector space consisting of all 10×10 real symmetric matrices $A = (a_{ij})$ such that $\displaystyle\sum_{j=1}^{10} a_{ij} = 0$ for all $i = 1, \ldots, 10$. Find the dimension of S.

Ans (a) ------------------ Ans (b) ------------------

8. Consider the vector space $\mathbb{R}^4$ with coordinates $(x_1, \ldots, x_4)$ and a symmetric form defined by

$$\langle (a_1, \ldots, a_4), (b_1, \ldots, b_4) \rangle = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

Find all vectors in $\mathbb{R}^4$ of norm 1 which are orthogonal, with respect to the above form, to the solution space of $x_1 - 2x_2 + 2x_3 + x_4 = 0$.

Ans --

9. Let A be a 2×2 matrix and I be the identity matrix. Assume that the null spaces of $A - 4I$ and $A - I$ respectively are spanned by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively. Find a matrix B such that $B^2 = A$.

Ans. –

10. Find (a) the minimal polynomial and (b) the characteristic polynomial of $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

Ans (a) – – – – – – – – – – – – – – – Ans (b) –

11. (a) Let $V = \{a_0 + a_1 x + \ldots + a_7 x^7 : a_i \in \mathbb{Q}\}$ be the vector space over $\mathbb{Q}$ consisting of polynomials in x of degree ≤ 7. Let $T : V \to V$ be the linear map defined by $T(p) = x^2 p'' - 6xp' + 12p$. Find the dimension of image of T.

(b) Let $T : \mathbb{R}^{21} \to \mathbb{R}^{21}$ be a linear transformation such that $T \circ T = 0$. Find the maximal possible rank of T.

Ans (a) – – – – – – – – – – – – – – – – – Ans (b) –

12. Consider the three matrices $P = \begin{pmatrix} 1 & 19 \\ 17 & 325 \end{pmatrix}$, $Q = \begin{pmatrix} 4 & 19 \\ 17 & 80 \end{pmatrix}$, $R = \begin{pmatrix} 18 & 19 \\ 17 & 18 \end{pmatrix}$.

(a) Which among P, Q and R belong to the set $\{A \mid x^T A x > 0,\ \forall x \neq 0 \in \mathbb{R}^2\}$?

(b) List all negative eigenvalues of P, Q and R.

Ans (a) – – – – – – – – – – – – – – – – – Ans (b) –

13. Let V be the real vector space of 2×2 matrices with a symmetric form defined by $\langle A, B\rangle = \operatorname{trace}(A^T B)$. Find the orthogonal projection, with respect to the above form, of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ to the subspace W consisting of all symmetric matrices.

Ans –

14. Let S be a largest possible set of 4×4 complex matrices such that each element of S has the set of eigenvalues as $\{1, 2\}$ and no two elements in S are similar. Find the number of elements in S.

Ans –

Real Analysis - 6 Questions

15. (a) Let $f : \mathbb{R} \to \mathbb{Q} \subset \mathbb{R}$ be a continuous function. Find $f'(1)$.

    (b) Let $f(x) = \begin{cases} \arctan(ax+b), & x < 0 \\ \frac{\pi}{4} e^{\sin bx}, & x \geq 0 \end{cases}$.

    Find all values of a and b such that f is differentiable.

Ans (a) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ Ans (b) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒

16. (a) Let f be a continuous function on $[0, 1]$ such that f is differentiable on $(0, 1)$, $f(0) = 0$ and $f(1) = 1$. Find all integers $n \geq 1$ such that there exist some $x_0 \in (0, 1)$ with $f'(x_0) = nx_0^{n-1}$.

    (b) Find all $r \in \mathbb{R}$ such that if f is any continuous function on $[1, 3]$ with $\int_1^3 f(x)\,dx = 1$, then there exist some $x_0 \in (1, 3)$ with $f(x_0) = r$.

Ans (a) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ Ans (b) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒

17. Let g be a bounded continuous function defined on $[0, 2\pi]$ such that $\int_0^{2\pi} g(x)\,dx = 1$. Suppose that $\int_0^{2\pi} p(x)g(x)\,dx = 0$ for all polynomials p with $p(\pi) = 0$. Evaluate $\int_0^{2\pi} e^{-2x} g(x)\,dx$.

Ans ‒

18. (a) Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{9n^2}$.

    (b) For positive integers n and m, evaluate $\int_0^1 x^m (\log x)^n\,dx$.

Ans (a) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ Ans (b) ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒

19. Find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$. You may use the Fourier cosine series of x^2 on $[0, \pi]$.

Ans ‒

20. Find the set of all $x \in \mathbb{R}$ such that the series $\frac{1}{(1-x)} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \ldots$ converges.

Ans ‒

Answers.

1. $\dfrac{5}{12}$
2. 2
3. $\dfrac{1}{2}$
4. $\dfrac{1}{48}$
5. $\dfrac{3}{5}$
6. 101
7. (a) 15, (b) 45
8. $\pm\dfrac{1}{2\sqrt{2}}(1,-2,2,-1)$
9. $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$.
10. (a) $(x+1)(x-3)$, (b) $(x+1)^3(x-3)$
11. (a) 6, (b) 10
12. (a) P and R (b) $42-\sqrt{1767}$
13. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
14. 10
15. (a) $f'(1)=0$, (b) $a=\pi/2$, $b=1$
16. (a) $\mathbb{N}$, (b) $1/2$
17. $e^{-2\pi}$
18. (a) $1/27$ (b) $\dfrac{(-1)^n n!}{(m+1)^{n+1}}$
19. $\dfrac{\pi^2}{12}$
20. $(-\infty,0)\cup[2,\infty)$

Registration number :

Name :

From the given list, choose the two subjects in which you are best prepared. Rank them by writing (1) and (2) against them.

In the interview, questions will be asked from these two subjects.

- Analysis (Complex Analysis, Functional Analysis, Measure Theory)
- Algebra (Group Theory, Ring Theory, Field Theory)
- Topology (Point set tolopogy, Algebraic topology)
- ODE and PDE
- Combinatorics
- Probability
- Statistics