Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (December 1, 2017)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks. Some of the questions have two parts of 1 mark each. There will be no partial credit.*
- *The answer to each question is a number (or a tuple of numbers), a set, or a function. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*
- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*

1. A coin is selected at random from a collection of 10 coins numbered 1 to 10 where the ith coin has probability $\frac{1}{2^i}$ for getting a head. The selected coin was tossed and resulted in a head. Find the probability that the selected coin was the coin numbered 1.

2. Let (X, Y) be uniform $(0,\ 1) \times (0, 1)$ random vector and $Z = \min\{X, Y\}$. Find $M(1)$, where $M(t)$ is the moment generating function of Z.

3. A woman leaves for work between 8 am and 8.30 am and takes between 40 to 50 minutes to get there. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at work before 9 am.

4. Let $X_1, \ldots, X_n$ be iid uniform$(0, \theta)$, where $\theta > 0$. Suppose $T_n = \max\{X_1, \ldots, X_n\}$. Then $f(x)$, the density of the limiting distribution of $n(1 - \frac{T_n}{\theta})$ as n goes to ∞, is

5. Let U, V, W be independent random variables with mean and variance both equal to 1. Let $X = U + V$ and $Y = V + W$. Then the covariance between X and Y is

6. Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables with mean 1. For $n \geq 1$ set
$$Y_n = \begin{cases} 1 & \text{if } X_n = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$
Then the constant to which $\frac{\sum_{i=1}^{n} Y_i}{n}$ converges with probability 1 is

7. (a) For what real values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^{n-1}}(x+2)^n$ converge?

   (b) What is the radius of convergence of the Taylor series of $\dfrac{2\tan x}{1+4x^2}$ around $x = 0$?

8. For any positive real numbers α and β, define

$$f(x) = \begin{cases} x^{\alpha} \sin\left(\frac{1}{x^{\beta}}\right) & , \text{ if } x \in (0,1], \\ 0 & , \text{ if } x = 0. \end{cases}$$

(a) For a given $\beta > 0$, find all values of α such that $f'(0)$ exists.

(b) For a given $\beta > 0$, find all values of α such that f is of bounded variation on $[0,1]$.

9. Let $\{a_n\}$ be a strictly increasing sequence of positive real numbers (i.e. $a_n < a_{n+1}$, for all natural number n) with $\lim_{n\to\infty} a_n = (\sqrt{2})^e$ and let $s_n = \sum_{k=1}^{n} a_k$. If $\{x_n\}$ is a strictly decreasing sequence of real numbers with $\lim_{n\to\infty} x_n = e^{\sqrt{2}}$, then find the value of

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{i=1}^{n} a_i x_i \,.$$

10. Let $\{f_n\}$ be a sequence of polynomials with real coefficients defined by $f_0 = 0$ and for $n = 0, 1, 2, \ldots$,

$$f_{n+1}(x) = f_n(x) + \frac{x^2 - f_n^2(x)}{2}.$$

Find $\lim_{n\to\infty} f_n$ on $[-1,1]$, where the limit is taken in the supremum norm of f_n over the interval $[-1,1]$.

11. (a) Find the number of connected components of the set

$$\left\{x \in \mathbb{R} : x^5 + 60x \geq 15x^3 + 10x^2 + 20\right\}.$$

(b) How many of the connected components in part (a) above are compact?

12. Let $P_n(x)$ be the Taylor polynomial for the exponential function, e^x, at $x = 0$. Compute the least n such that $|e - P_n(1)| < 10^{-4}$.

13. Consider $\mathbb{R}^3$ (column vectors) with the standard inner product. Let L be the one dimensional subspace of $\mathbb{R}^3$ spanned by the column vector $(2, 1, 2)^t$. Let A be the 3×3 matrix such that the linear transformation of $\mathbb{R}^3$ given by $x \mapsto Ax$ is orthogonal projection onto the line L. Then the sum of the entries of A equals

14. For $\alpha \in \mathbb{R}$, let $q(x_1, x_2) = x_1^2 + 2\alpha x_1 x_2 + \frac{1}{2} x_2^2$, for $(x_1, x_2) \in \mathbb{R}^2$.

    (a) Take $\alpha = 1/4$ and let B be the symmetric matrix of q with respect to the basis $\{(1, 0), (1, 1)\}$ of $\mathbb{R}^2$. Then the entry in row 1, column 2 of B equals

    (b) Find all values of α for which the signature of q is 1.

15. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Let

$$\mathcal{A} := \left\{ A \in M_8(\mathbb{R}) \;\middle|\; \begin{array}{l} \text{the characteristic polynomial of } A \text{ is } (x-1)^2(x-2)^6 \\ \text{and the minimal polynomial of } A \text{ is } (x-1)(x-2)^2. \end{array} \right\}$$

and

$$G := \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a matrix } A \in \mathcal{A} \text{ for which} \\ k \text{ is the geometric multiplicity of the eigenvalue 2} \end{array} \right\}.$$

Then the set G equals

16. Consider the vector space $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) equipped with the inner product

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = 2(a_1 b_1 + a_2 b_2 + a_3 b_3) - (a_1 b_2 + a_2 b_1 + a_2 b_3 + a_3 b_2).$$

Write down all vectors in $\mathbb{R}^3$ which are orthogonal to the plane $x_1 - 2x_2 + 2x_3 = 0$ and have norm 1.

17. Let A be a 10×10 matrix defined by $A = (a_{ij})$ where $a_{ij} = 1 - (-1)^{i-j}$. If $P(x)$ is the minimal polynomial of A then

(a) what is the degree of $P(x)$?

(b) what is the coefficient of x in $P(x)$?

18. Let A be a diagonal matrix whose characteristic polynomial is

$$P(x) = (x-15)(x-14)^2(x-13)^3 \cdots (x-2)^{14}(x-1)^{15}.$$

Let V be the set of all 120×120 matrices commuting with A. Then the dimension of V is

19. For each real number α, let B_α be the bilinear form

$$B_\alpha\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) =$$

$$x_1x_2 + 2y_1y_2 + 11z_1z_2 + (\alpha+1)(x_1y_2 + y_1x_2) + 3(x_1z_2 + z_1x_2) + (\alpha+2)(y_1z_2 + z_1y_2).$$

Find the set $\{\alpha \in \mathbb{R} : B_\alpha \text{ is positive definite}\}$.

20. Let A and B be $(2017)^2 \times (2017)^2$ complex matrices with A being invertible. Find the maximum number of complex numbers α for which the matrix $\alpha A + B$ is not invertible.